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Exact Results for a Generalized Classical $O(n)$ Matrix Spin Model

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A generalized $O(n)$ matrix version of the classical Heisenberg model, introduced by Fuller and Lenard as a classical limit of a quantum model, is solved exactly in one dimension. The free energy is analytic and the pair correlation functions decay exponentially for all finite temperatures. It is shown, however, that even for a finite number of spins the model has a phase transition in the $n \to \infty$ limit. The transition features a specific heat jump, zero long-range order at all temperatures, and zero correlation length at the critical point. The Curie–Weiss version of the model is also solved exactly and shown to have standard mean-field type behavior for all finite n and to differ from the one-dimensional results in the $n \to \infty$ limit.

KEY WORDS: Heisenberg model; classical limit; matrix models; phase transition; spherical limit; mean-field model.

1. INTRODUCTION

The problem of the classical limit of quantum spin systems has been the subject of some interesting papers,[1-3] beginning with the work of Lieb[1] who showed that if one considers a quantum system of spins, each having angular momentum l, and one normalizes the spins by dividing by l, then in the (classical) limit $l \to \infty$, the normalized quantum partition function becomes that of the corresponding classical system of spins which are three-dimensional unit vectors, and the trace in the quantum partition function is replaced by an integration for each spin over the unit sphere S^2 in $\mathbb{R}^3$. In particular the classical limit of the quantum Heisenberg model is the classical Heisenberg ($O(3)$) model.

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The classical Heisenberg model on S^2 has a straightforward generalization to the so-called n-vector model where the spins are n-dimensional unit vectors and the configuration space is S^{n-1}. This model has the interesting property that it is equivalent to the spherical model in the limit $n \to \infty$.[4,5,6]

Related to Lieb's work an interesting question then naturally arises: is there any quantum correspondent of the n-vector model? The answer to this question has been provided in the negative by Simon.[2] That is, there is no quantum analog of the usual n-vector model. Hence in looking for an $O(n)$ generalization of the classical Heisenberg model, subject to the condition that it is the classical limit of some quantum spin model in the sense of Lieb, one has to discover what replaces S^2 as the classical limit space.

A direction in $\mathbb{R}^3$ is given by a unit vector, but equally well by the plane orthogonal to it. Therefore S^2 may be thought of as the manifold of oriented planes through the origin; that is, the Grassmann manifold $G(2, 3)$. Thus, a possible candidate for a generalized classical manifold is $G(2, n)$, the manifold of oriented two-planes in $\mathbb{R}^n$, passing through the origin. This simple intuitive idea was the starting point of the paper by Fuller and Lenard,[3] who introduced a class of $O(n)$ matrix spin models, replacing the n-vector model as a generalization of the $O(3)$ classical Heisenberg model. Moreover, they showed that their classical model can be obtained from a corresponding quantum model in the limit of infinite angular momentum.

Our purpose here is to study certain instances of the Fuller–Lenard model which can be solved exactly.

In the following section we present a formulation of the Fuller–Lenard model and in Section 3 we give an exact solution of the one-dimensional model with nearest-neighbor interactions. Pair correlations for the one-dimensional model are evaluated in Section 4 and in Section 5 we study the $n \to \infty$ spherical limit of the one-dimensional model. In this limit we find unexpectedly that the model has a phase transition of a rather peculiar type: the specific heat has a jump discontinuity at a finite critical temperature but there is zero long-range order for all finite temperatures. Moreover, one does not need the thermodynamic limit to obtain a phase transition in the $n \to \infty$ limit. Thus in the spherical limit even a two-spin system has a phase transition!

In order to show that the model is not dimension independent in the spherical limit we examine the Curie–Weiss version of the model in Section 6. Our results are summarized and discussed in the final section.

2. FORMULATION OF THE FULLER–LENARD MODEL

We begin by characterizing the model configuration space $G(2, n)$ which, as we have seen, is a possible classical limit space generalization to higher dimensional spaces of S^2.

An oriented two-plane σ in $\mathbb{R}^n$ can be defined as the equivalence class with respect to transformations of the form

$$\begin{aligned} \mathbf{a}' &= \mathbf{a} \cos\theta + \mathbf{b} \sin\theta \\ \mathbf{b}' &= -\mathbf{a} \sin\theta + \mathbf{b} \cos\theta \end{aligned} \tag{2.1}$$

of ordered pairs $(\mathbf{a}, \mathbf{b})$ of two orthogonal unit vectors in $\mathbb{R}^n$. A function $f = f(\mathbf{a}, \mathbf{b})$ may be regarded as defined on $G(2, n)$ whenever (2.1) implies $f(\mathbf{a}, \mathbf{b}) = f(\mathbf{a}', \mathbf{b}')$ and in such a case we write $f = f(\sigma)$.

Alternatively, $G(2, n)$ can be viewed as the set

$$M_n = \{m \in \mathcal{M}_{n\times n}(\mathbb{R}) \mid m^t = -m, \operatorname{Tr}(m^t m) = 2, \operatorname{rank} m = 2\} \qquad n \geqslant 2 \tag{2.2}$$

where $\mathcal{M}_{n\times n}(\mathbb{R})$ is the set of real $n \times n$ matrices and the superscript t denotes the transpose of the matrix. In order to see that this is so we observe that $m(\sigma) \in M_n$ can be expressed uniquely in terms of the vectors $(\mathbf{a}, \mathbf{b})$ determining σ by

$$m_{jk}(\sigma) = a_j b_k - a_k b_j \tag{2.3}$$

Expressed in this form, it is clear that when $n = 3$ the right-hand side of (2.3) represents the vector product of $\mathbf{a}$ and $\mathbf{b}$ and the nonzero elements of $m(\sigma)$ are the components of the unit vector orthogonal to the plane generated by $\mathbf{a}$ and $\mathbf{b}$.

It should be noted that $O(n)$ acts transitively on $G(2, n)$ (or M_n) and hence $G(2, n)$ (or M_n) is a homogeneous space of $O(n)$. It follows that the invariant integral over $G(2, n)$ (or M_n) can be defined in terms of the Haar integral over $O(n)$ by

$$\int_{G(2,n)} f(\sigma)\, d\mu(\sigma) = \int_{O(n)} f(g\sigma_0)\, dg \tag{2.4}$$

where dg is the normalized Haar measure on $O(n)$ and $\sigma_0 \in G(2, n)$ is an arbitrary fixed plane. Alternatively, in terms of M_n, we have

$$\int_{M_n} f(m)\, d\mu(m) = \int_{O(n)} f(g m_0 g^t)\, dg \tag{2.5}$$

where $m_0 \in M_n$ is arbitrary.

To construct their quantum model, Fuller and Lenard considered the sequence of spherical harmonic representations of $O(n)$ and introduced generalized spin operators L_{jk} as the infinitesimal generators of the representation, the Hilbert space on which they act being the representation space. Defining next the quantum spin Hamiltonian and its classical correspondent in much the same way as in Lieb's original work, Fuller and Lenard were then able to show that the classical model, with $G(2, n)$ as its limit manifold, could be obtained from the quantum model as an appropriate limit of infinite angular momentum. It is to be noted, however, that $G(2, n)$ is not the only possible generalization of S^2. The classical limit space in fact depends on which representation one chooses for $O(n)$[2].

The class of quantum spin models considered by Fuller and Lenard includes the generalized Heisenberg model defined by

$$\mathscr{H}_Q = -\sum_{r,r'=1}^{N} J_{rr'} \sum_{j,k=1}^{n} L_{jk}^{(r)} L_{jk}^{(r')} \tag{2.6}$$

where r, r' denote lattice sites. The classical limit of this model, which will be our concern in this paper, has Hamiltonian given by

$$\mathscr{H}_{cl} = -\frac{1}{2} \sum_{r,r'=1}^{N} J_{rr'} \operatorname{Tr}(m_r m_{r'}^t) \tag{2.7}$$

with $m_r \in M_n$, $r = 1, 2, \ldots, N$.

Although the trace in (2.7) can be viewed as a scalar product it is better to think of this model as having matrices assigned to each site rather than vectors in the n-vector model. Notice, however, that in view of the restrictions (2.2) imposed on the matrices in question, the dimension of the underlying configuration space is $2n-4$. It should also be noted that when $n=3$ and 2, the model reduces to the classical Heisenberg and Ising models, respectively. We also note in passing that similar matrix models appear in field theory[7-10] in connection with the so-called "planar-diagram" approximation.

3. PARTITION FUNCTION OF THE ONE-DIMENSIONAL MODEL WITH NEAREST NEIGHBOR INTERACTIONS

The Hamiltonian for the one-dimensional model with nearest-neighbor interactions only is given by

$$\mathscr{H} = -J \sum_{r=1}^{N-1} \frac{1}{2} \operatorname{Tr}(m_r^t m_{r+1}) \tag{3.1}$$

where m_r are elements of M_n, the set of $n \times n$ skew-symmetric matrices of rank two satisfying the normalization condition

$$\tfrac{1}{2}\operatorname{Tr}(m_r^t m_r) = 1 \qquad r = 1, 2, \ldots, N \tag{3.2}$$

and the t superscript in (3.1) and (3.2) denotes transpose of the matrix.

The partition function of the model is given by

$$Z_{n,N}(K) = \int_{M_n} \cdots \int_{M_n} \exp\left[K \sum_{r=1}^{N-1} \frac{1}{2} \operatorname{Tr}(m_r^t m_{r+1})\right] \prod_{r=1}^{N} d\mu(m_r) \tag{3.3}$$

where $K = J/kT$, and $d\mu(m)$ denotes the probability measure defined on M_n which is invariant under $O(n)$. That is

$$\int_{M_n} f(m)\, d\mu(m) = \int_{O(n)} f(g m_0 g^t)\, dg \tag{3.4}$$

where the integral on the right-hand side is taken with respect to the normalized Haar measure over $O(n)$ and m_0 is an arbitrary element of M_n.

In order to evaluate the partition function (3.3) we need to evaluate integrals of the form

$$\begin{aligned} I_r &= \int_{M_n} \exp\left[\frac{K}{2} \operatorname{Tr}(m_r^t m_{r+1})\right] d\mu(m_{r+1}) \\ &= \int_{O(n)} \exp\left[\frac{K}{2} \operatorname{Tr}(m_r^t g_{r+1} m_0 g_{r+1}^t)\right] dg_{r+1} \end{aligned} \tag{3.5}$$

for some fixed (arbitrary) $m_0 \in M_n$. Since $O(n)$ acts transitively on M_n, there exists a rotation $s_r \in O(n)$ such that

$$m_r = s_r m_0 s_r^t \tag{3.6}$$

Taking account of the invariance of the measure with respect to $O(n)$, it then follows on defining $g = s_r^t g_{r+1}$, that

$$I_r = \int_{O(n)} \exp\left[\frac{K}{2} \operatorname{Tr}(m_0^t g m_0 g^t)\right] dg \equiv \lambda_n(K) \tag{3.7}$$

is independent of r and, in particular on referring back to (3.5), independent of m_r.

By integrating successively over m_N, $m_{N-1}, \ldots, m_2$ in the expression (3.3) for the partition function and using the above results we then have that

$$Z_{n,N}(K) = [\lambda_n(K)]^{N-1} \tag{3.8}$$

It is interesting to note that the factorization property (3.8) for the open chain model, which is well known for the Ising and n-vector model chains, is manifest in general here as a group invariance property of the underlying configuration space.

One can proceed in several ways to evaluate the integral (3.7). Here we use a direct group integration method which is of some intrinsic pedagogical interest. In the following section we use a different method which is conceptually simpler and a more direct and convenient computational method, particularly for the evaluation of correlation functions.

To proceed with the group integration method we first express the integral over the (doubly connected) group $O(n)$ in terms of integrals over $SO(n)$, which in our special case, give

$$\lambda_n(K) = \tfrac{1}{2}\{\tilde{\lambda}_n(K) + \tilde{\lambda}_n[(-1)^{n+1} K]\} \tag{3.9}$$

where $\tilde{\lambda}_n(K)$ is given by (3.7) but with the integral now over $SO(n)$ instead of $O(n)$.

The normalized invariant measure on $SO(n)$ is given in terms of Euler angles by

$$dg = \prod_{l=1}^{n} 2\pi^{-l/2}\Gamma(l/2) \prod_{k=1}^{n-1} \prod_{j=1}^{k} (\sin\theta_j^{(k)})^{j-1}\, d\theta_j^{(k)} \tag{3.10}$$

where

$$\theta_1^{(k)} \in [0, 2\pi) \qquad \text{and} \qquad \theta_j^{(k)} \in [0, \pi) \qquad j > 1 \tag{3.11}$$

and in general any element of $SO(n)$ can be written as a product of proper planar rotations taken through the set of Euler angles.

Finally, to simplify the computation we choose for the arbitrary matrix m_0 in (3.7) the particular form

$$m_0 = \left(\begin{array}{c|cc} O_{n-2} & \multicolumn{2}{c}{0} \\ \hline & 0 & 1 \\ 0 & -1 & 0 \end{array}\right) \tag{3.12}$$

where O_k denotes the $k \times k$ null matrix.

As an example, the Ising case $n = 2$ has

$$g = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{3.13}$$

and

$$\tfrac{1}{2}\,\mathrm{Tr}(m_0^t\, g m_0\, g^t) = 1 \tag{3.14}$$

yielding from (3.7) and (3.9) the well-known result

$$\lambda_2(K) = \frac{1}{2}\int_{SO(2)} (e^K + e^{-K})\, dg = \cosh K \tag{3.15}$$

In the Heisenberg case $n = 3$ we have

$$\lambda_3(K) = \frac{1}{16\pi^2}\iint_0^{2\pi} d\theta\, d\phi \int_0^{\pi} \sin\psi\, d\psi\, \exp[K(\cos\theta\cos\phi - \sin\theta\sin\phi\cos\psi)]$$

$$= K^{-1}\sinh K \tag{3.16}$$

In general when $n \geqslant 4$ the choice (3.12) for m_0 reduces the complexity of the problem considerably since in this case for arbitrary g,

$$\tfrac{1}{2}\,\mathrm{Tr}(m_0^t\, g m_0\, g^t) = \cos\theta\cos\phi - \sin\theta\sin\phi\cos\theta'\cos\phi' \tag{3.17}$$

where

$$\theta = \theta_{n-1}^{(n-1)} \qquad \phi = \theta_{n-2}^{(n-1)} \qquad \theta' = \theta_{n-2}^{(n-2)} \qquad \text{and} \qquad \phi' = \theta_{n-3}^{(n-3)}$$

Using the expression (3.10) for dg and integrating over all $\theta_j^{(k)}$ except for those appearing in (3.17) one then obtains for $n > 3$

$$\lambda_n(K) = \frac{(n-2)(n-3)}{(2\pi)^2}\iiiint_0^{\pi} \exp[K\cos\theta\cos\phi - \sin\theta\sin\phi\cos\theta'\cos\phi']$$

$$\cdot \sin^{n-2}\theta(\sin\theta\sin\theta')^{n-3}\sin^{n-4}\phi'\, d\theta\, d\phi\, d\theta'\, d\phi' \tag{3.18}$$

Finally, after integrating successively over ϕ', θ', ϕ, and θ one obtains, as shown in Appendix A, the expression

$$\lambda_n(K) = {}_1F_2\left(1; \frac{n}{2}, \frac{n-1}{2}; \frac{K^2}{4}\right) \tag{3.19}$$

where ${}_1F_2$ is the generalized hypergeometric function defined by

$${}_1F_2(\alpha; \beta, \gamma; z) = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{(\beta)_k(\gamma)_k}\cdot\frac{z^k}{k!} \tag{3.20}$$

with

$$(\alpha)_0 = 1 \qquad \text{and} \qquad (\alpha)_k = \alpha(\alpha+1)\cdots(\alpha+k-1) \tag{3.21}$$

Although (3.19) was derived under the assumption $n > 3$ it in fact reproduces the Ising and Heisenberg results given above when n equals two and three, respectively. Further special cases are

$$\lambda_4(K) = 2!\, K^{-2}(\cosh K - 1) \tag{3.22}$$

and

$$\lambda_5(K) = 3!\, K^{-3}(\sinh K - K) \tag{3.23}$$

In general $\lambda_n(K)$, which may be thought of as the partition function per site, is an analytic function of K for all finite n. We will see in Section 5, however, that in the $n \to \infty$ limit one obtains a phase transition which in fact is already present when the system consists of merely two interacting spins.

4. THE PAIR CORRELATION FUNCTION IN ONE DIMENSION

In order to calculate correlation functions it is convenient to parametrize the configuration space in terms of pairs of mutually orthogonal n-dimensional unit vectors $\sigma_r = (\mathbf{a}_r, \mathbf{b}_r)$ for each site. This can be done by noting that an $n \times n$ skew-symmetric matrix of rank two satisfying the constraint (3.2) can be expressed in the form

$$m_{ij} = a_i b_j - a_j b_i, \qquad i, j = 1, 2,..., n \tag{4.1}$$

if and only if

$$\|\mathbf{a}\| = \|\mathbf{b}\| = 1 \qquad \text{and} \qquad \mathbf{a} \cdot \mathbf{b} = 0 \tag{4.2}$$

With this parametrization the invariant measure on M_n can be expressed as

$$d\mu(\sigma) = C_n \delta(\|\mathbf{a}\|^2 - 1)\, \delta(\|\mathbf{b}\|^2 - 1)\, \delta(\mathbf{a} \cdot \mathbf{b})\, d\mathbf{a}\, d\mathbf{b} \tag{4.3}$$

where $\delta(x)$ is the Dirac delta function, $d\mathbf{a}$ and $d\mathbf{b}$ are the usual Lebesque measures on $\mathbb{R}^n$, and

$$C_n = \pi^{(1/2) - n}\, \Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{n-1}{2}\right) \tag{4.4}$$

is such that the measure is properly normalized.

To evaluate the integral (3.7), written now in the form

$$\lambda_n(K; Y) = \int \exp[(K/2) \operatorname{Tr}(Y^t m(\sigma))]\, d\mu(\sigma) \tag{4.5}$$

we use (4.3) and the integral representation

$$\delta(x)=\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{px}\,dp \tag{4.6}$$

to express (4.5) in terms of multidimensional Gaussian integrals. After interchanging orders of integration and performing the Gaussian integrals one obtains, as shown in Appendix B, the expression

$$\lambda_n(K;\,Y)=\pi^{1/2}\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n-1}{2}\right)(2\pi i)^{-3}\int_{\alpha-i\infty}^{\alpha+i\infty} dp_1\,dp_2\,dp_3\,e^{p_1+p_2}$$
$$\cdot\left\{\mathrm{Det}\left[\left(p_1p_2-\frac{1}{4}p_3{}^2\right)I_n+\frac{1}{4}K^2Y^2\right]\right\}^{-1/2} \tag{4.7}$$

where I_n is the $n\times n$ unit matrix and α is chosen so that all singularities in the integrand are to the left of the line $z=\alpha$ in the complex plane. The result (4.7) actually holds for any skew-symmetric $n\times n$ matrix Y.

Using the fact that $m(\sigma')\in M_n$ can be expressed in the form $gm_0\,g^t$ with m_0 given by (3.12), the determinant in (4.7) with $Y=m(\sigma')$ can be seen to have the value

$$(p_1p_2-\tfrac{1}{4}p_3{}^2)^{n-2}[(p_1p_2-\tfrac{1}{4}p_3{}^2)-(K^2/4)]^2 \tag{4.8}$$

Subsequent contour integration and some perseverance reduces (4.7) to the result (3.19) found by group integration.

Our main purpose in this section is to evaluate the pair correlation function defined by

$$\left\langle\frac{1}{2}\mathrm{Tr}(m_i^tm_j)\right\rangle=[\lambda_n(K)]^{-(N-1)}\int_{M_n}\cdots\int_{M_n}\frac{1}{2}\mathrm{Tr}(m_i^tm_j)$$
$$\cdot\exp\left[K\sum_{r=1}^{N-1}\frac{1}{2}\mathrm{Tr}(m_r^tm_{r+1})\right]\prod_{r=1}^{N}d\mu(m_r) \tag{4.9}$$

In order to do this we need to evaluate the integral

$$I(m')=\int_{M_n}d\mu(\sigma)\,m(\sigma)\exp\left\{\frac{K}{2}\mathrm{Tr}[m'^tm(\sigma)]\right\}\qquad m'\in M_n \tag{4.10}$$

which, from (4.5), can be written as

$$I(m')=\frac{2}{K}\left(\frac{\partial}{\partial Y_{kl}}[\lambda_n(K;\,Y)]\right)_{Y=m'} \tag{4.11}$$

Using the expression (4.7) and the result

$$\frac{\partial}{\partial Y_{kl}} \{\mathrm{Det}[\alpha^2 I + \gamma^2 Y^2]\}^{-1/2} \bigg|_{Y = m' \in M_n} = \frac{\gamma^2 m'_{kl}}{(\alpha^2 - \gamma^2)[\mathrm{Det}(\alpha^2 I + \gamma^2 m'^2)]^{1/2}} \tag{4.12}$$

gives

$$\begin{aligned} I(m') &= \frac{K}{2} m' \int_{\alpha - i\infty}^{\alpha + i\infty} dp_1\, dp_2\, dp_3\, e^{p_1 + p_2} \\ &\quad \cdot \left(p_1 p_2 - \frac{1}{4} p_3^2\right)^{n/2 - 1} \left(p_1 p_2 - \frac{1}{4} p_3^2 - \frac{K^2}{4}\right)^{-2} \\ &= m' \frac{d}{dK} [\lambda_n(K)] \end{aligned} \tag{4.13}$$

which shows, among other things, that m is an eigenfunction of the transfer integral operator with kernel $\exp[K/2\, \mathrm{Tr}(m'tm)]$ with eigenvalue $d\lambda_n/dK$, the largest eigenvalue being λ_n (with a constant as eigenfunction).

From (4.9) and (4.13), it is not difficult to show that the pair correlation function is given by

$$\left\langle \frac{1}{2} \mathrm{Tr}(m_r^t m_{r'}) \right\rangle = \left(\lambda_n^{-1} \frac{d}{dK} \lambda_n\right)^{|r - r'|} \tag{4.14}$$

which is precisely of the form for the usual n-vector model, showing exponential decay for all finite n and K.

5. THE SPHERICAL LIMIT IN ONE DIMENSION

In order to examine the behavior of the model in the limit $n \to \infty$ (with K scaled by a factor of n) we use the integral representations

$$\begin{aligned} {}_1F_2(s; s + \mu, \nu + 1, a) &= \frac{\Gamma(\nu + 1)}{B(\mu, s)} \int_0^1 x^{s-1} (ax)^{-\nu/2} (1 - x)^{\mu - 1} \\ &\quad \cdot I_\nu(2a^{1/2} x^{1/2})\, dx \end{aligned} \tag{5.1}$$

and

$$I_\nu(z) = \frac{(z/2)^n}{\Gamma(\nu + \frac{1}{2})\, \Gamma(\frac{1}{2})} \int_{-1}^{1} (1 - t^2)^{\nu - (1/2)} e^{zt}\, dt \tag{5.2}$$

to write $\lambda_n(nK)$, Eq. (3.19), in the form

$$\lambda_n(nK) = \frac{2\Gamma(n/2)}{\Gamma[(n/2)-1]\,\Gamma(\frac{1}{2})} \int_0^1 \int_{-1}^1 x(1-x^2)^{-2}(1-y^2)^{-2} \exp[nf(x, y)] \tag{5.3}$$

where

$$f(x, y) = Kxy + \tfrac{1}{2}\log(1-x^2)(1-y^2) \tag{5.4}$$

When $K<1$, $f(x, y)$ takes on its maximum value when $x=y=0$ while for $K>1$ the maximum is obtained when $x=y=(1-K^{-1})^{1/2}$. It then follows from Laplace's method that the free energy $\psi(K)$ in the limit $n\to\infty$ is given by

$$-\beta\psi(K) = \lim_{n\to\infty} n^{-1}\log\lambda_n(nK) = \begin{cases} K-1-\log K & K>1 \\ 0 & K<1 \end{cases} \tag{5.5}$$

In order to evaluate the pair correlation function (4.14) we need to evaluate the correction term in the asymptotic expansion of the integral (5.3). Straightforward analysis gives

$$\lambda_n^{-1}\frac{d}{dK}\lambda_n\bigg|_{K\to Kn} \sim \begin{cases} n^{-1}C(K)(1-K)^{-2} & K<1 \\ 1-K^{-1} & K>1 \end{cases} \quad \text{as} \quad n\to\infty \tag{5.6}$$

where $C(K)$ is analytic at $K=1$, and from (4.14) we then have

$$\lim_{n\to\infty}\left\{\left\langle\frac{1}{2}\,\mathrm{Tr}(m_r^t m_{r'})\right\rangle_{K\to Kn}\right\} = \begin{cases} \delta_{r,r'} & K<1 \\ (1-K^{-1})^{|r-r'|} & K>1 \end{cases} \tag{5.7}$$

Thus, in the "spherical limit" ($n\to\infty$) the model exhibits a rather unusual type of phase transition. The "correlation length" at the critical point $K_c=1$ becomes zero and there is zero long-range order at all finite temperatures. The specific heat, however, has a jump discontinuity at T_c, taking the values k when $K>1$ and zero when $K<1$. It is to be noted that the occurrence of the phase transition is independent of the size of the system so that even the "two-spin" or zero-dimensional model undergoes a phase transition. It is interesting, in fact, that the expression (5.5) is identical with the corresponding expression for the zero-dimensional and classical spherical models[11] but that the correlation functions behave differently in these cases. Finally, we observe that this unusual type of phase transition is not generated by the usual mechanism of asymptotic eigenvalue degeneracy of the transfer operator, where one might have expected the left-hand side of (5.6) to approach unity for $K>1$.

To show that the model is not dimension independent we consider in the following section the spherical limit of the Curie–Weiss version of the model.

6. THE CURIE–WEISS MODEL AND THE SPHERICAL LIMIT

The Hamiltonian for the Curie–Weiss model consisting of N spins is given by

$$\mathscr{H} = -\frac{J}{2N} \sum_{r,r'=1}^{N} \operatorname{Tr}(m_r^t m_{r'}) = -\frac{J}{2N} \operatorname{Tr}(A^t A) \tag{6.1}$$

where

$$A = \sum_{r=1}^{N} m_r \tag{6.2}$$

is $n \times n$ skew symmetric and $m_r \in M_n$.

Using the Gaussian integral identity

$$e^{h^2/4\alpha} = \sqrt{\alpha/\pi} \int_{-\infty}^{\infty} dx\, e^{-\alpha x^2 + hx} \tag{6.3}$$

the partition function for the model can be written as

$$\begin{aligned} Z_{n,N}(K) &= \int \prod_r d\mu(\sigma_r) \exp\left[\frac{K}{2N} \operatorname{Tr}(A^t A)\right] \\ &= \left(\sqrt{\frac{NK}{4\pi}}\right)^{n(n-1)/2} \int_{-\infty}^{\infty} \prod_{i<j} dy_{ij} \exp\left[-\frac{NK}{8} \operatorname{Tr}(Y^t Y)\right] \\ &\quad \cdot \left\{\int_{M_n} d\mu(m) \exp\left[\frac{K}{2} \operatorname{Tr}(Y^t m)\right]\right\}^N \end{aligned} \tag{6.4}$$

where $K = \beta J$ and Y is the $n \times n$ skew-symmetric matrix whose elements are the integration variables y_{ij} when $i < j$ and $-y_{ij}$ when $i > j$.

In the thermodynamic limit $N \to \infty$, with n fixed, the integrals over the y_{ij} can be evaluated by Laplace's method yielding for the free energy

$$\begin{aligned} \beta\psi_n(K) &= -\lim_{N \to \infty} N^{-1} \log Z_{n,N}(K) \\ &= \min_{\substack{Y \\ (Y^t = -Y)}} \left\{\frac{K}{8} \operatorname{Tr}(Y^t Y) - \log \lambda_n(K; Y)\right\} \end{aligned} \tag{6.5}$$

where use has been made of the definition (4.5) for the group integral appearing in (6.4).

Assuming that Y has rank $2k$ and nonzero eigenvalues $\pm ic_\alpha$, $\alpha = 1, 2, \ldots, k$, the representation (4.7) for $\lambda_n(K; Y)$ becomes

$$\lambda_n(K; Y) = \frac{C_n}{(2\pi i)^3} \int_{\alpha - i\infty}^{\alpha + i\infty} dp_1\, dp_2\, dp_3\, e^{p_1 + p_2} \left[p_1 p_2 - \frac{1}{4} p_3^2 \right]^{(2k-n)/2} \cdot \prod_{\alpha=1}^{k} \left[p_1 p_2 - \frac{1}{4} p_3^2 - \left(\frac{K}{2} c_\alpha \right)^2 \right]^{-1} \tag{6.6}$$

where

$$C_n = \pi^{(1/2)-n} \Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{n-1}{2}\right) \tag{6.7}$$

Expanding out the product in (6.6) and performing the integrals over p_3, p_2, and p_1, we obtain

$$\begin{aligned} \lambda_n(K; Y) &= C_n \sum_{l_1 \cdots l_k} \left[\Gamma\left(\frac{n}{2} + \sum l_\alpha\right) \Gamma\left(\frac{n-1}{2} + \sum l_\alpha\right) \right]^{-1} \prod_\alpha (Kc_\alpha/2)^{2l_\alpha} \\ &= \sum_{l=0}^{\infty} \frac{(K^2u^2/4)^l}{(n/2)_l\, [(n-1)/2]_l} \left[\sum_{\substack{l_1 \cdots l_k \\ (\sum l_\alpha = l)}} x_1^{l_1} x_2^{l_2} \cdots x_k^{l_k} \right] \end{aligned} \tag{6.8}$$

where

$$u^2 \equiv \sum_{\alpha=1}^{k} c_\alpha^2 = \frac{1}{2} \operatorname{Tr}(Y^t Y) \tag{6.9}$$

and

$$x_\alpha = c_\alpha^2/u^2 \tag{6.10}$$

Using the fact that $x_1 + x_2 + \cdots + x_k = 1$, it is not difficult to see that

$$\sum_{\substack{l_1 \cdots l_k \\ (\sum l_\alpha = l)}} \prod_{i=1}^{k} x_i^{l_i} \leqslant 1 \tag{6.11}$$

and then to prove by induction on k that equality in (6.11) is realized only when one of the x_i's is unity and the remainder are zero. That is, from (3.19) and (3.20)

$$\lambda_n(K; Y) \leqslant {}_1F_2\left(1; \frac{n}{2}, \frac{n-1}{2}; \frac{K^2u^2}{4}\right) \tag{6.12}$$

with equality realized on rank two skew-symmetric matrices satisfying (6.9) with $k=1$. Equation (6.5) then shows that

$$\beta\psi_n(K)=\min_{u\geqslant 0}\left\{\frac{Ku^2}{4}-\log {}_1F_2\left(1;\frac{n}{2},\frac{n-1}{2};\frac{K^2u^2}{4}\right)\right\} \tag{6.13}$$

which is of standard mean-field form for all finite n.

In the spherical limit we obtain, using (5.5), the expression

$$\beta\psi(K)=\lim_{n\to\infty} n^{-1}\beta\psi_n(nK)=\min_{u\geqslant 0}\left\{\frac{Ku^2}{4}-g(uK)\right\} \tag{6.14}$$

where

$$g(x)=\begin{cases} x-1-\log x & x\geqslant 1 \\ 0 & x<1 \end{cases} \tag{6.15}$$

Performing the minimization in (6.14) then gives

$$\beta\psi(K)=\begin{cases} \frac{1}{2}-\frac{1}{2}(K+\sqrt{K^2-2K})+\log(K+\sqrt{K^2-2K}) & K>K_c \\ 0 & K\leqslant K_c \end{cases} \tag{6.16}$$

where K_c is the unique solution of the equation

$$1-(K+\sqrt{K^2-2K})+\log(K+\sqrt{K^2-2K})^2=0 \qquad K>2 \tag{6.17}$$

7. DISCUSSION

In this paper we have studied the classical $O(n)$ matrix spin model introduced by Fuller and Lenard. Exact results were obtained for the free energy and pair correlation function for the open chain. As expected the model shows analytic behavior for all finite n and temperature. In the spherical limit $n\to\infty$, however, the one-dimensional model has an unusual kind of phase transition in which the specific heat has a jump discontinuity at a finite critical temperature but with zero long-range order for all finite temperatures. The phase transition is, in fact, manifest for finite systems so that even a two-spin system has the same critical behavior as the infinite system in the spherical limit.

We also obtained an exact expression for the free energy of the Curie–Weiss version of the model which has the standard mean-field behavior for all finite n. In the spherical limit one also obtains mean-field behavior showing that in this limit the model is not dimension independent.

The general behavior of the model in the spherical limit is an open question at this time but from the behavior of the one-dimensional model we expect that the model will have richer critical phenomena in this limit than the usual spherical model which is obtained from an $n \to \infty$ limit of the usual n-vector model.

Finally, we note that a similar model to the one we considered here is currently being investigated in ϕ^4 field theory in connection with the planar diagram approximation.(7–10) Also, integration over a link in the Feynmann path integral for $U(n)$ lattice QCD(12) requires the calculation of similar integrals to those considered in this article. The $n \to \infty$ limit is also studied in these papers, so perhaps our spherical limit results have some relevance to this work.

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APPENDIX A. EVALUATION OF (3.18)

In order to evaluate the integral (3.18) we use the result

$$\int_0^\pi e^{z\cos\theta} \sin^{2\nu}\theta \, d\theta = \sqrt{\pi}\,(z/2)^{-\nu}\,\Gamma\left(\nu+\frac{1}{2}\right) I_\nu(z) \tag{A.1}$$

to first perform the integral over ϕ'. The integral over ϕ can then be performed using the fact that

$$\begin{aligned}\int_0^{\pi/2} &(\sin\phi)^{[(n-4)/2]+1} \cosh(K\cos\theta\cos\phi)(Kx\sin\theta)^{(n-4)/2} \\ &\cdot I_{(n-4)/2}(Kx\sin\theta\sin\phi)\,d\phi \\ &= \sqrt{\frac{\pi}{2}}\,(K\sqrt{x^2\sin^2\theta+\cos^2\theta})^{-(n-3)/2}\, I_{(n-3)/2}(K\sqrt{x^2\sin^2\theta+\cos^2\theta})\end{aligned} \tag{A.2}$$

With $x=\sin\theta'$ and $y=\cos\theta$ we then have

$$\begin{aligned}\lambda_n(K) &= \pi^{-1}(n-2)(n-3)\Gamma\left(\frac{n-3}{2}\right) \\ &\cdot \iint_0^1 dx\,dy(1-y^2)^{1/2}[(1-x^2)(1-y^2)]^{(n-4)/2} \\ &\cdot (K\sqrt{x^2+y^2-x^2y^2}/2)^{(n-3)/2}\, I_{(n-3)/2}(K\sqrt{x^2+y^2-x^2y^2})\end{aligned} \tag{A.3}$$

The elliptic substitution

$$\left.\begin{array}{l} u=\sqrt{x^2+y^2-x^2y^2} \\ v=f(x,y) \end{array}\right\} u\in(0,1), \quad v\in(0,\pi/2) \tag{A.4}$$

such that

$$(1-y^2)^{1/2}\,dx\,dy=u\,du\,dv \tag{A.5}$$

then gives

$$\begin{aligned} \lambda_n(K) &= \frac{2\Gamma[(n-1)/2]}{B[(n-2)/2,1]}\int_0^1 u(1-u^2)^{(n-4)/2}\, I_{(n-3)/2}(Ku)(Ku/2)^{-(n-3)/2}\,du \\ &= {}_1F_2\left(1;\frac{n}{2},\frac{n-1}{2};\frac{K^2}{4}\right) \end{aligned} \tag{A.6}$$

where in the last step we have used (5.1).

APPENDIX B. EVALUATION OF (4.5)

In order to evaluate (4.5) we use the normalized measure (4.3) and the representation (4.6) for the three delta functions appearing in (4.3) to write (4.5) in the form

$$\begin{aligned} \lambda_n(K;Y) = {} & \pi^{(1/2)-n}\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n-1}{2}\right)(2\pi i)^{-3}\int_{\alpha-i\infty}^{\alpha+i\infty} dp_1\,dp_2\,dp_3\,e^{p_1+p_2} \\ & \cdot\int_{\mathbb{R}^{2n}} \exp[-(x,Tx)]\,dx \end{aligned} \tag{B.1}$$

where x denotes the $2n$-dimensional vector (**ab**), T is the $2n\times 2n$ matrix

$$T=\begin{pmatrix} p_1 I_n & \frac{1}{2}(p_3 I_n - KY) \\ \frac{1}{2}(p_3 I_n + KY) & p_2 I_n \end{pmatrix} \tag{B.2}$$

and, in the above reduction, use has been made of the representation (4.1) of $m(\sigma)$ in terms of **a** and **b**.

On performing the Gaussian integral in (B.1) we are left with

$$\begin{aligned} \lambda_n(K;Y) = {} & \pi^{1/2}\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n-1}{2}\right)(2\pi i)^{-3} \\ & \cdot\int_{\alpha-i\infty}^{\alpha+i\infty} dp_1\,dp_2\,dp_3\,e^{p_1+p_2}(\mathrm{Det}\,T)^{-1/2} \end{aligned} \tag{B.3}$$

which is easily seen, from (B.2), to reduce to the required result given in (4.7).

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